2D Fourier Transform

Definition
\[ F(u, v) = \mathcal{F}\{f(x, y)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{i2\pi(ux+vy)} \, dx \, dy \]
where \( x, y \) are rectangular spatial coordinates and \( u, v \) are spatial frequencies.

We can rewrite this as two 1D transforms:
\[ F(u, v) = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(x, y) e^{i2\pi ux} \, dx \right] e^{i2\pi vy} \, dy \]
i.e. for each \( y \) take a 1D FT with respect to \( x \), then take a 1D FT with respect to \( y \).

Polar Coordinates

Since
\[ ux + vy = \rho r (\cos \phi \cos \theta + \sin \phi \sin \theta) = \rho r \cos(\phi - \theta), \]
then the Fourier transform in polar coordinates is
\[ F(\rho, \phi) = \mathcal{F}\left\{f(r, \theta)\right\} = \int_{0}^{2\pi} \int_{0}^{\infty} f(r, \theta) e^{i2\pi \rho r \cos(\phi - \theta)} r \, dr \, d\theta \]
and the inverse transform is
\[ f(r, \theta) = \mathcal{F}^{-1}\{F(\rho, \phi)\} = \int_{0}^{2\pi} \int_{0}^{\infty} F(\rho, \phi) e^{-i2\pi \rho r \cos(\phi - \theta)} \rho \, d\rho \, d\phi \]

Separability in Polar Coordinates
A function \( f(r, \theta) \) is separable in polar coordinates if it can be written in the form
\[ f(r, \theta) = f_r(r) f_\theta(\theta). \]
Suppose such an \( f \) is circularly symmetric, with \( f_\theta(\theta) = 1 \). As a specific example, we will consider a 2D circular (or cylinder) function:
\[
circ(r) = \begin{cases} 
1 & r < 1 \\
1/2 & r = 1 \\
0 & r > 1 
\end{cases}
\]
The Fourier transform of a circularly symmetric function is

\[ F(\rho, \phi) = 2\pi \int_0^\infty r f_r(r) J_0(2\pi \rho r) \, dr . \]

This is also known as the Hankel transform of order zero and as the Fourier-Bessel transform. The function \( J_0 \) is the zero order Bessel function of the first kind defined as

\[ J_0(a) = \frac{1}{2\pi} \int_0^{2\pi} e^{ia \cos(\theta-\phi)} \, d\theta . \]

It oscillates like a damped cosine.

Continuing with our specific example, the Fourier transform of \( \text{circ}(r) \) is

\[ \mathcal{F}\{\text{circ}(r)\} = 2\pi \int_0^\infty r \text{circ}(r) J_0(2\pi \rho r) \, dr = 2\pi \int_0^1 r J_0(2\pi \rho r) \, dr . \]

Substitute \( r' = 2\pi \rho r \), and \( dr' = 2\pi \rho \, dr \) to find:

\[ \mathcal{F}\{\text{circ}(r)\} = \frac{1}{2\pi \rho^2} \int_0^{2\pi \rho} r' J_0(r') \, dr' = \frac{J_1(2\pi \rho)}{\rho} , \quad \text{since } \int_0^\infty x J_0(x) \, dx = \alpha J_1(\alpha) \]

where \( J_1 \) is the first order Bessel function of the first kind, similar to a damped sinusoid.

The function \( \text{somb}(\rho) \) or \( \text{sombrero} \) (also known as Mexican hat, Bessinc, and jinc) is defined as

\[ \text{somb}(\rho) = \frac{2J_1(2\pi\rho)}{2\pi\rho} , \]

and is pictured on the right side below. Thus, the Fourier transform of \( \text{circ}(r) \) is proportional to a sombrero function of \( \rho \), the radial coordinate in frequency space:

\[ \mathcal{F}\{\text{circ}(r)\} = \pi \text{somb}(\rho) . \]